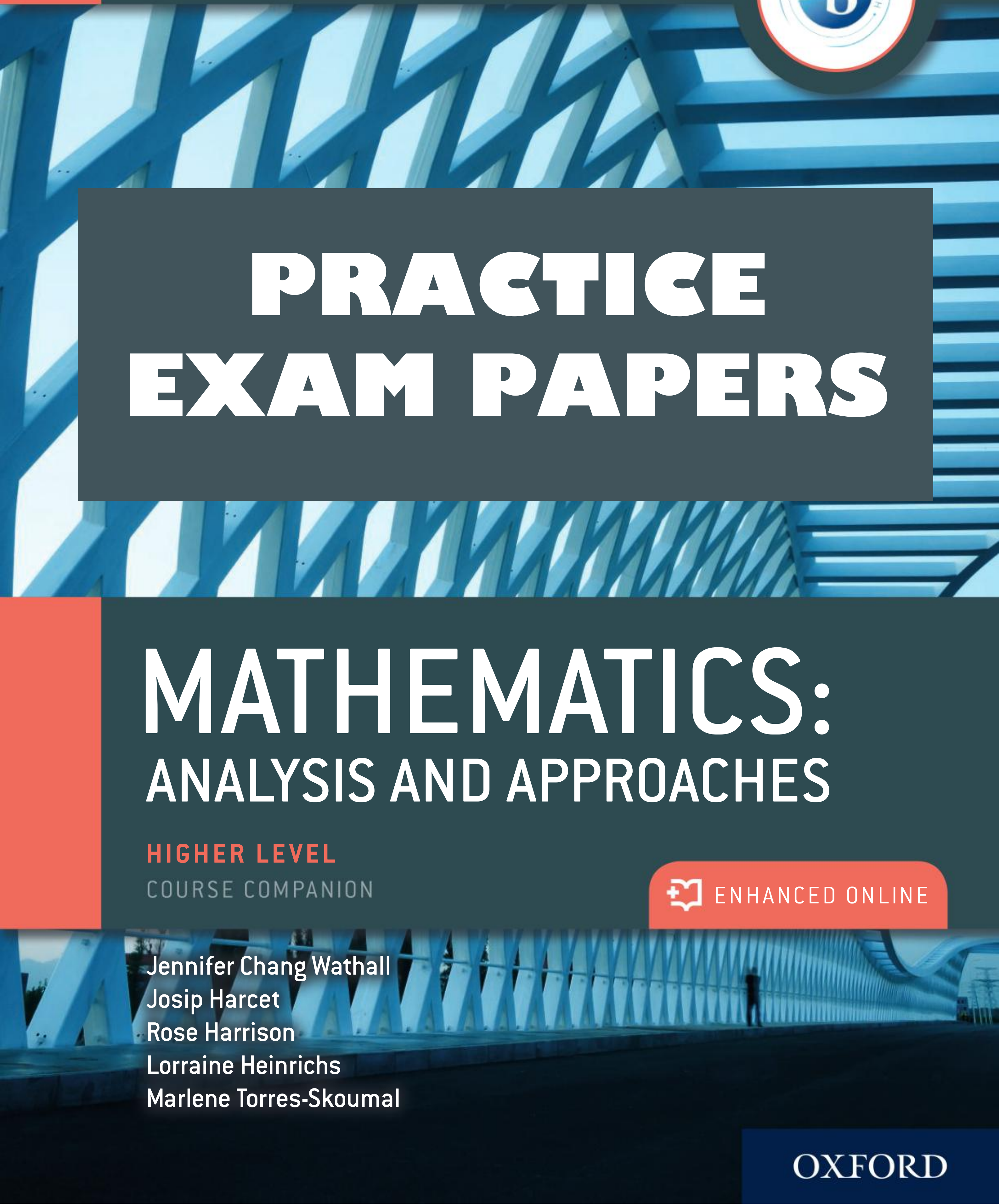
OXFORD IB DIPLOMA PROGRAMME



PRACTICE EXAM PAPERS

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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Paper 1

Time allowed: 2 hours

Maximum number of marks: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You are not allowed to use a calculator for this paper.

Section A

1 [Maximum mark: 5]

a Write down the number of elements in each of the following sets:

i $\varnothing$ **ii** $\{\varnothing, \{\varnothing\}\}$ **iii** $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$. [3]

b **i** Write down the next set in this sequence of sets.

ii State how many elements in this set are members of the set of Natural numbers. [2]

2 [Maximum mark: 7]

The random variable X satisfies the $N(\mu, \sigma^2)$ distribution. It is known that $P(X \le 7) = 0.3$.

a Find $P(X > 7)$. [1]

b Let $Z = \dfrac{X - \mu}{\sigma}$.

i State the distribution that Z will satisfy.

ii Find $P\left(Z \le \dfrac{7 - \mu}{\sigma}\right)$. [3]

c Two independent readings of X are to be taken. Calculate the probability that exactly one of these readings is less than or equal to 7. [3]

3 [Maximum mark: 5]

A set of data, with values given in ascending order, is as follows:

$2, 3, 4, 6, 7, 9, 10, 11, 14, 17, x$.

a Given that x is an outlier, find an inequality that x must satisfy. [3]

b If this outlier is removed, state how the mean will change (without calculating it). [1]

c In general, if outliers are removed from a set of data, state what will happen to the standard deviation. [1]

4 [Maximum mark: 5]

Find the constant term in the expansion of $\left(x+\frac{2}{x}\right)^4$. [5]

5 [Maximum mark: 8]

Let $a=\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $b=\begin{pmatrix}1\\y\\2\end{pmatrix}$.

a If vectors *a* and *b* are parallel, find the value of *y*. [2]

b If vectors *a* and *b* are perpendicular, find the value of *y*. [3]

c If $|b| = 3$ find the possible values of *y*. [3]

6 [Maximum mark: 9]

Using integration by parts twice, find $\int x(\ln x)^2 dx$. [9]

7 [Maximum mark: 7]

A polynomial is given by $P(x) = x^3 - 7x + c$.

When $P(x)$ is divided by $x-1$, the remainder is -12.

a Find the value of the constant *c*. [2]

b Factorize $P(x)$ as far as possible. [5]

8 [Maximum mark: 9]

By repeated use of l'Hôpital's rule and justifying its use at each stage, determine the value of $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$. [9]

Section B

9 [Maximum mark: 11]

a Solve the equation $\cos^2 x + \frac{3}{2}\sin x - \frac{3}{2} = 0$, for $0^\circ \le x \le 360^\circ$. [7]

b Without using calculus, explain why $\cos^2 x + \frac{3}{2}\sin x$ is always strictly smaller than $\frac{5}{2}$. [4]

10 [Maximum mark: 10]

For two sets A and B, the following information is given:

$\mathrm{P}(A) = 0.25$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(A' \cap B') = 0.45$

a Determine, with a reason, whether A and B are mutually exclusive. [5]

b Determine, with a reason, whether A and B are independent. [3]

c Find $\mathrm{P}\left((A \cap B) | A\right)$. [2]

11 [Maximum mark:19]

a Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x$ given that $y(0) = 2$. Give your answer in the form $y = y(x)$ for some function of x to be determined. [8]

b By repeatedly differentiating each term of the differential equation given in part **a**, find the first four terms in the Maclaurin expansion of $y(x)$, noting that this can involve terms that are zero. [7]

c **i** Use the Maclaurin expansion for e^x, along with your function $y = y(x)$ from part **a**, to find the first four **non-zero** terms in the Maclaurin expansion of $y(x)$. This should confirm your answer in part **b**.

ii State the general term in the Maclaurin expansion of $y(x)$, involving x^{2r} for $r \ge 1$. [4]

12 [Maximum mark:15]

The probability density function for the $N(\mu, \sigma^2)$ distribution is given by $f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

a Find the z-value of the stationary point of $f(z)$. [4]

b Classify the nature of this turning point and hence find the mode of this distribution. [3]

c Explain why $f(z)$ is symmetrical about $z = \mu$ and hence find the mean of this distribution. [2]

d Find the z-values of any points of inflection of $f(z)$. You must clearly justify that these are inflection points. Use your justification to comment on the concavity of $f(z)$. [6]

Markscheme

Section A

1 a i 1 A1

ii 2 A1

iii 3 A1

[3 marks]

b i $\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\{\{\varnothing\}\}\}\}$ A1

ii none A1

[2 marks]

[Total: 5 marks]

2 a $1-0.3=0.7$ A1

[1 mark]

b i $N\left(0,1^2\right)$ A1 A1

ii 0.3 A1

[3 marks]

c Need $X_1>7$ and $X_2<7$, or vice versa. R1

Probability is $2\times0.7\times0.3=0.42$ M1 A1

[3 marks]

[Total: 7 marks]

3 a IQR = 14-4=10 A1

so x>14+1.5×10⇒x>29 M1 A1

[3 marks]

b The mean will decrease. R1

[1 mark]

c The standard deviation will decrease. R1

[1 mark]

[Total: 5 marks]

4 General term is $\binom{4}{r}x^{4-r}\left(\frac{2}{x}\right)^r=\binom{4}{r}x^{4-2r}\left(2^r\right)$ (M1)

For constant term, we require $4-2r=0\Rightarrow r=2$ (R1) A1

Constant term is $\binom{4}{2}x^2\left(\frac{2}{x}\right)^2=6\times4=24$ A1 A1

[5 marks]

[Total: 5 marks]

5 a $b=\lambda a\Rightarrow\lambda=\frac{1}{2}\Rightarrow y=\frac{1}{2}$ (M1) A1

[2 marks]

b $a\cdot b=\begin{pmatrix}2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}1\\y\\2\end{pmatrix}=2+y+8=0$ for perpendicular vectors M1 A1

$\Rightarrow y=-10$ A1

[3 marks]

c $\sqrt{1+y^2+4}=3\Rightarrow y^2=4\Rightarrow y=\pm2$ M1 A1 A1

[3 marks]

[Total: 8 marks]

6 $\int x(\ln x)^2 dx$

Let $u = (\ln x)^2$ and $\dfrac{dv}{dx} = x$		M1
$\dfrac{du}{dx} = \dfrac{2}{x}\ln x \qquad v = \dfrac{x^2}{2}$	A1	A1
$\int x(\ln x)^2 dx = \dfrac{x^2}{2}(\ln x)^2 - \int x\ln x\, dx$		A1
Let $u = \ln x$ and $\dfrac{dv}{dx} = x$		M1
$\dfrac{du}{dx} = \dfrac{1}{x} \qquad v = \dfrac{x^2}{2}$	A1	A1
$\int x(\ln x)^2 dx = \dfrac{x^2}{2}(\ln x)^2 - \left(\dfrac{x^2}{2}\ln x - \int \dfrac{x}{2}dx\right)$		A1
$= \dfrac{x^2}{2}(\ln x)^2 - \dfrac{x^2}{2}\ln x + \dfrac{x^2}{4} + c$		A1

[9 marks]
[Total: 9 marks]

7 a $P(1) = -12 \Rightarrow 1 - 7 + c = -12 \Rightarrow c = -6$	M1	A1

[2 marks]

b $P(x) = x^3 - 7x - 6$		
$P(-1) = 0 \qquad x+1$ is a factor	M1	A1
$P(-2) = 0 \qquad x+2$ is a factor	M1	A1
(Alternatively, you might spot that $x-3$ is a factor)		
Factorizes as $(x+1)(x+2)(x-3)$		A1

[5 marks]
[Total: 7 marks]

8 $\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3}$ is an indeterminate expression of the form $\dfrac{0}{0}$, so l'Hôpital's rule applies.		R1
$\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{\sec^2 x - \cos x}{3x^2}$ (which is still of the form $\dfrac{0}{0}$)	M1	A1
$= \lim\limits_{x\to 0}\dfrac{2\sec^2 x\tan x + \sin x}{6x}$ (which is still of the form $\dfrac{0}{0}$)	A1	A1
$= \lim\limits_{x\to 0}\dfrac{4\sec^2 x\tan^2 x + 2\sec^4 x + \cos x}{6}$	A2	A1
$= \dfrac{3}{6} = \dfrac{1}{2}$		A1

[9 marks]
[Total: 9 marks]

Section B

9 a $\cos^2 x + \dfrac{3}{2}\sin x - \dfrac{3}{2} = 0 \Rightarrow 1 - \sin^2 x + \dfrac{3}{2}\sin x - \dfrac{3}{2} = 0 \Rightarrow \sin^2 x - \dfrac{3}{2}\sin x + \dfrac{1}{2} = 0$		M1	A1
$\Rightarrow \left(\sin x - \dfrac{1}{2}\right)(\sin x - 1) = 0 \Rightarrow \sin x = \dfrac{1}{2}$ or 1		M1	A1
$x = 30^\circ, 150^\circ$ or 90°	A1	A1	A1

[7 marks]

b Since $\sin x$ and $\cos x$ are both between -1 and $+1$, $\cos^2 x \le 1$ and $\frac{3}{2}\sin x \le \frac{3}{2}$ R1 A1

Hence $\cos^2 x + \frac{3}{2}\sin x \le \frac{5}{2}$ A1

and since $\cos x = \pm 1$ and $\sin x = 1$ cannot happen for the same value of x R1

then $\cos^2 x + \frac{3}{2}\sin x < \frac{5}{2}$ AG

[4 marks]
[Total: 11 marks]

10 a $P(A' \cap B') = 0.45 \Rightarrow P((A \cup B)') = 0.45 \Rightarrow P(A \cup B) = 0.55$ M1 A1

Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $\quad 0.55 = 0.25 + 0.4 - P(A \cap B) \Rightarrow P(A \cap B) = 0.1$

M1 A1

$P(A \cap B) \ne 0$ so they are not mutually exclusive. R1

[5 marks]

b $P(A) \times P(B) = 0.25 \times 0.4 = 0.1$ M1 A1

$P(A \cap B) = P(A) \times P(B)$ so they are independent R1

[3 marks]

c $P((A \cap B)|A) = \frac{P(A \cap B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.25} = 0.4$ M1 A1

[2 marks]
[Total: 10 marks]

11 a Integrating factor is $e^{\int x\,dx} = e^{\frac{x^2}{2}}$ M1 A1

$$e^{\frac{x^2}{2}}\frac{dy}{dx} + xe^{\frac{x^2}{2}}y = xe^{\frac{x^2}{2}} \Rightarrow \frac{d\left(e^{\frac{x^2}{2}}y\right)}{dx} = xe^{\frac{x^2}{2}}$$ M1

$ye^{\frac{x^2}{2}} = \int xe^{\frac{x^2}{2}}\,dx = e^{\frac{x^2}{2}} + c$ M1 A1

$y(0) = 2 \Rightarrow 2 = 1 + c \Rightarrow c = 1$ M1 A1

$ye^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} + 1$

$y = 1 + e^{-\frac{x^2}{2}}$ A1

[8 marks]

b Using $y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y'''(0)\frac{x^3}{3!} + \ldots$

$y(0) = 2$

$y'(0) + 0 = 0 \Rightarrow y'(0) = 0$ M1 A1

Differentiating each term of the differential equation gives:
$y''(x) + y(x) + xy'(x) = 1 \Rightarrow y''(0) + y(0) + 0 = 1 \Rightarrow y''(0) = -1$ M1 A1

Differentiating again gives:
$y'''(x) + y'(x) + y'(x) + xy''(x) = 0 \Rightarrow y'''(0) + 2y'(0) + 0 = 0 \Rightarrow y'''(0) = 0$ M1 A1

$y(x) = 2 + 0x - \frac{1}{2}x^2 + 0x^3 \ldots$ A1

[7 marks]

c i Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$$y(x) = 1 + 1 - \frac{x^2}{2} + \frac{\left(\frac{x^4}{4}\right)}{2!} - \frac{\left(\frac{x^6}{8}\right)}{3!} \ldots$$ M1 A1

$= 2 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$ (agreeing with previous answer) A1

ii Required general term is $\frac{(-1)^r x^{2r}}{2^r r!}, r \geq 1$ A1

[4 marks]
[Total: 19 marks]

12 a $f'(z) = \frac{-1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \frac{(z-\mu)}{\sigma^2} = 0 \Rightarrow z = \mu$ M1 A1 R1 A1

[4 marks]

$z > \mu \Rightarrow f'(z) < 0$ and $z < \mu \Rightarrow f'(z) > 0$ giving a max at $z = \mu$ R1 A1

Hence, mode is μ. A1

[3 marks]

Replacing $(z-\mu)$ with $(\mu - z)$ in $f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$ leaves the function unaltered, so is symmetrical about $z = \mu$. R1

Hence mean is μ. A1

[2 marks]

$f''(z) = \frac{-1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \left(\frac{1}{\sigma^2} - \frac{(z-\mu)^2}{\sigma^4} \right) = 0 \Rightarrow (z-\mu)^2 = \sigma^2 \Rightarrow z = \mu \pm \sigma$ M1 A1 R1 A1

The sign of $f''(z)$ is given by the table

z	$z < \mu - \sigma$	$z = \mu - \sigma$	$\mu - \sigma < z < \mu + \sigma$	$z = \mu + \sigma$	$z > \mu - \sigma$
$f''(z)$	+ve	0	−ve	0	+ve

Confirming that we do have 2 points of inflection, as the sign of $f''(z)$ changes. R1

Concavity goes concave up, concave down, concave up, changing at $\mu - \sigma$ and $\mu + \sigma$. A1

[6 marks]
[Total: 15 marks]

Paper 2

Time allowed: 2 hours

Maximum number of marks: 110 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphic display calculator for this paper.

Section A

1 [Maximum mark: 8]

An arithmetic progression has a second term of 5 and an eleventh term of 41.

a Find the first term and the common difference. [3]

b **i** Find and simplify a formula for the nth term, u_n .

ii Hence find the value of n for which $u_n = 549$. [2]

c **i** Find and simplify a formula for the sum of the first n terms, S_n .

ii Hence find the smallest value of n for which $S_n > 1000$. [3]

2 [Maximum mark: 8]

a Write down the value of $\int_0^1 x^2\sqrt{x^3+1}\,dx$, giving your answer to 4 decimal places. [3]

b **i** Find $\int x^2\sqrt{x^3+1}\,dx$.

ii Hence find the **exact** value of $\int_0^1 x^2\sqrt{x^3+1}\,dx$, simplifying your answer as far as possible.

[6]

3 [Maximum mark: 8]

a If $y = 5x^4$, find a linear expression connecting $\log y$ and $\log x$. [3]

b By taking logarithms to base 10 of both sides, solve the equation $3^{4x} = 5^{2x-1}$.

Give your answer as *both*

i an exact form, and **ii** correct to 4 significant figures. [5]

4 [Maximum mark: 7]

Find the acute angle, in degrees, between the planes $x+3y+2z=0$ and $x-y+5z=0$. [7]

5 [Maximum mark: 7]

The probability distribution of a continuous random variable, X, is given by

$$f(x)=\begin{cases}\lambda e^{x} & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}.$$

a Calculate the numerical value of λ correct to 3 s.f. [3]

b Find the mean of X. Give your answer in the form λA, where A is a constant to be expressed in exact form. [4]

6 [Maximum mark: 8]

Let $z=e^{\frac{i\pi}{30}}$.

a Find the smallest positive integer, n, so that z^n is a purely imaginary number. [5]

b Find the smallest positive integer, n, so that z^n is a purely real, positive number. [3]

7 [Maximum mark: 9]

Let X be a random variable, where $E(X)=5$, $Var(X)=2$

The random variable $W=3X+28$.

a Find

i $E(W)$ **ii** $Var(W)$. [4]

b Find $E(W^2)$. [3]

c If W is normally distributed, find $P(W>50)$. [2]

Section B

8 [Maximum mark: 11]

Paired bivariate data (x,y) is collected from 11 students, where x is their time to swim 100 m (measured in seconds) and y is their time to run 200 m (also measured in seconds). The data is given in the following table:

x	100	81	120	104	180	200	152	102	94	131	142
y	40	35	44	39	51	60	48	40	37	43	47

a Calculate the Pearson product moment correlation coefficient (r) for this data, and state what this value of r implies about the relationship between the swimming and running times. [5]

b **i** Calculate the equation of the linear regression line of y on x.

ii Write down the mean point $(\bar{x}, \bar{y})$ that the linear regression line of y on x must pass through. [4]

c State two reasons why the equation found in part b i should not be used to estimate the swimming time of a student with a running time of 23 seconds. [2]

9 [Maximum mark: 14]

A rational function is given by $y(x) = \dfrac{x-3}{4x+2}, x \neq -\dfrac{1}{2}$.

a Find $\dfrac{dy}{dx}$ and hence show that the graph of $y(x)$ is always increasing. [3]

b Find $\dfrac{d^2y}{dx^2}$ and hence comment on the concavity of the graph of $y(x)$. [4]

c For the graph of $y(x)$, write down the equations of

i the vertical asymptote

ii the horizontal asymptote. [2]

d For the graph of $y(x)$, write down the equations of

i the coordinates of the x-axis intercept

ii the coordinates of the y-axis intercept. [2]

e Sketch the graph of this function, showing the information that has been obtained in parts **a** to **e**. [3]

10 [Maximum mark: 12]

Consider the system of equations

$$x + 2y + 3z = 7$$
$$2x + y - z = 2$$
$$3x + 3y + pz = q$$

a Solve these equations for

i $p = 4, \ q = 13$

ii $p = 2, \ q = 4$

iii $p = 2, \ q = 9$ [10]

b By using your results from parts (a)(ii) and (iii), rather than by direct calculation, state the value of $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{vmatrix}$ and justify your answer. [2]

11 [Maximum mark: 18]

Let $\mathbf{a} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \cos\beta \\ \sin\beta \\ 0 \end{pmatrix}$.

a By calculating the scalar product of $\mathbf{a}$ and $\mathbf{b}$, prove the identity
$\cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$. [4]

b By calculating the vector product of $\mathbf{a}$ and $\mathbf{b}$, prove the identity
$\sin(\alpha - \beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta$. [7]

c Use parts **a** and **b** and properties of trigonometrical functions, that should be stated, to obtain similar expressions for (i) $\cos(\alpha + \beta)$ and (ii) $\sin(\alpha + \beta)$. [7]

Markscheme
Section A

1 a $a+d=5, \quad a+10d=41$ M1

solving gives $a=1,\ d=4$ A1 A1

[3 Marks]

b i $u_n = 1+(n-1)4 = 4n-3$ A1

ii solving $4n-3=549$ gives $n=138$ A1

[2 marks]

c i $S_n = \frac{n}{2}(2+(n-1)4) = 2n^2 - n$ A1

ii solving (e.g. with "table" on GDC) $2n^2 - n > 1000$ gives $n=23$ ($S_n = 1035$) M1 A1

[3 marks]
[Total: 8 marks]

2 a 0.4063 (4 d.p.) A2

[2 marks]

b i by inspection or substitution $\frac{2}{9}(x^3+1)^{\frac{3}{2}}+c$ (M1) A2

ii $\left(\frac{2}{9}(2)^{\frac{3}{2}}\right)-\left(\frac{2}{9}\right)=\frac{4\sqrt{2}-2}{9}$ M1 A1 A1

[6 marks]
[Total: 8 marks]

3 a $\log y = \log(5x^4) = \log 5 + \log x^4$ M1 A1

So $\log y = \log 5 + 4\log x$ A1

[3 marks]

b $\log(3^{4x}) = \log(5^{2x-1}) \Rightarrow 4x\log 3 = (2x-1)\log 5$ M1 A1

$\Rightarrow (4\log 3 - 2\log 5)x = -\log 5$ A1

i $x = \frac{\log 5}{2\log 5 - 4\log 3}$ A1

ii -1.369 (4 s.f.) A1

[5 marks]
[Total: 8 marks]

4 Angle between planes will is equal to the angle between the normal to the planes (R1)

That is, the acute angle between $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\5\end{pmatrix}$. A1 A1

Using the dot product, $\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\5\end{pmatrix}=\left|\begin{pmatrix}1\\3\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\5\end{pmatrix}\right|\cos\theta$ M1 A1

$8=\sqrt{14}\sqrt{27}\cos\theta \Rightarrow \theta = 65.7^\circ\ (3sf)$ A1 A1

[7 marks]
[Total: 7 marks]

5 a Require $\int_0^1 \lambda e^{x^2}\,dx = 1 \Rightarrow \lambda = 0.684$ (3 s.f.) M1 A2

[3 marks]

b $\mu = \int_0^1 \lambda x e^{x^2}\,dx$ M1 A1

$= \lambda \left[\frac{e^{x^2}}{2} \right]_0^1 = \lambda \frac{(e-1)}{2}$ A1 A1

[4 marks]
[Total: 7 marks]

6 a $z^n = 3^n e^{\frac{in\pi}{30}}$ M1 A1

For z^n to be purely imaginary, require $\frac{n\pi}{30}$ to be an odd multiple of $\frac{\pi}{2}$ R1

$\frac{n}{30} = \frac{2k+1}{2} \Rightarrow n = 15(2k+1)$, smallest value is 15 M1 A1

[5 marks]

b For z^n to be purely real and positive, require $\frac{n\pi}{30}$ to be a multiple of 2π R1

$\frac{n}{30} = 2k \Rightarrow n = 60k$, smallest value is 60 M1 A1

[3 marks]
[Total: 8 marks]

7 a i $E(W) = 3 \times 5 + 28 = 43$ M1 A1

ii $Var(W) = 3^2 \times 2 = 18$ M1 A1

[4 marks]

b $Var(W) = E(W^2) - (E(W))^2 \Rightarrow E(W^2) = 18 + 43^2 = 1867$ M1 A1 A1

[3 marks]

c $W \sim N(43, 18)$, $P(W > 50) = 0.0495$ (3 s.f.) M1 A1

[2 marks]

[Total: 9 marks]

Section B

8 a $r = 0.979$ (3 s.f.) A2

Strong, positive, linear correlation A1 A1 A1

[5 marks]

b i $y = 0.187x + 20.1$ A1 A1

ii $(128, 44)$ M1 A1

[4 marks]

c The line of x on y should be used instead when estimating a swimming time from a running time; R1

Using the line to estimate a swimming time when the running time is 23 seconds would be extrapolation a long way away from the given data. R1

[2 marks]
[Total: 12 marks]

9 a $\frac{dy}{dx} = \frac{(4x+2)-(x-3)4}{(4x+2)^2} = \frac{14}{(4x+2)^2}$ M1 A1

Since $\frac{14}{(4x+2)^2} > 0$ for all x, so the graph of $y(x)$ is always increasing. R1

[3 marks]

b $\frac{d^2y}{dx^2} = \frac{-112}{(4x+2)^3}$ M1 A1

If $x > -\frac{1}{2}$ then $(4x+2)^3 > 0$ and hence $\frac{d^2y}{dx^2} = \frac{-112}{(4x+2)^3} < 0$, so the graph is concave down. R1

If $x < -\frac{1}{2}$ then $(4x+2)^3 < 0$ and hence $\frac{d^2y}{dx^2} = \frac{-112}{(4x+2)^3} > 0$,

so the graph is concave up. R1

[4 marks]

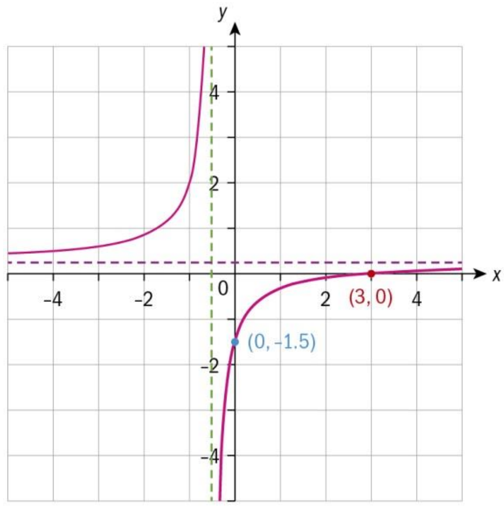
c **i** $x = -\frac{1}{2}$ **ii** $y = \frac{1}{4}$ A1 A1

[2 marks]

d **i** $(3, 0)$ **ii** $\left(0, -\frac{3}{2}\right)$ A1 A1

[2 marks]

e



A1 for $y(x)$; A1 for asymptotes; A1 for axes intercepts

[3 marks]

[Total: 14 marks]

10 a **i** $x = \frac{7}{3}, y = -\frac{2}{3}, z = 2$ (M1) A1 A1 A1

ii no solutions (M1) A1

iii $x = -1 + \frac{5}{3}\lambda, y = 4 - \frac{7}{3}\lambda, z = \lambda$ (M1) A1 A1 A1

[10 marks]

b Det $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 0$, since if non-zero then the inverse matrix would exist and there would be exactly one solution. A1 R1

[2 marks]

[Total: 12 marks]

11 a $a \cdot b = \cos\alpha\cos\beta + \sin\alpha\sin\beta + 0$ M1 A1

Also $a \cdot b = |a||b|\cos(\alpha - \beta) = \cos(\alpha - \beta)$ M1 A1

Giving $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ AG

[4 marks]

b $a \times b = \begin{pmatrix} 0 \\ 0 \\ \cos\alpha\sin\beta - \sin\alpha\cos\beta \end{pmatrix}$ M1 A1

Also $a \times b = |a||b|\sin(\alpha - \beta)\hat{n}$ M1 A1

Equating the modulus of the two above vectors gives $\sin(\alpha-\beta)=\pm(\cos\alpha\sin\beta-\sin\alpha\cos\beta)$

M1 A1

Taking $\beta=0$ shows that it must be the negative sign that is taken R1

Hence $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ AG

[7 marks]

c **i** Replacing β with $-\beta$ in part **a** gives M1

$\cos(\alpha+\beta)=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$ A1

So since $\cos x$ is an even function and $\sin x$ is an odd function, we have R1

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ A1

ii Replacing β with $-\beta$ in part **b** gives M1

$\sin(\alpha+\beta)=\sin\alpha\cos(-\beta)-\cos\alpha\sin(-\beta)$ A1

So since $\cos x$ is an even function and $\sin x$ is an odd function, we have

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ A1

[7 marks]

[Total: 18 marks]

Paper 3

Time allowed: 1 hour

Maximum number of marks: 60 marks

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphic display calculator for this paper.

1 [Maximum mark: 26]

In this question, you will attempt to find information about the sums of sequences that are neither arithmetic nor geometric progressions.

a **i** Let $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Use partial fractions to write $\frac{1}{k(k+1)}$ as the sum of two fractions, both with linear denominators. Hence show that $S_n = \frac{n}{n+1}$.

ii Let $S = \lim_{n\to\infty} S_n$. Write down the value of S. [8]

b **i** Let $T_n = \sum_{k=1}^{n} \frac{1}{k(k+2)}$. Use a similar method to that in part (a) to find and simplify an expression for T_n .

ii Let $T = \lim_{n\to\infty} T_n$. Write down the value of T. [10]

c **i** Let $Q_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Explain why $\sum_{i=2}^{n} \frac{1}{i^2} < \sum_{i=2}^{n} \frac{1}{(i-1)i}$ and hence show that

$$Q_n < 1 + S_{n-1} = \frac{2n-1}{n}$$

ii Let $Q = \lim_{n\to\infty} Q_n$. Show that $\frac{49}{36} < Q \leq 2$. [8]

2 [Maximum mark: 34]

In this question, you will investigate the conditions required for a volume of revolution to have the same volume when the curve is rotated about either the x-axis or the y-axis.

Consider the portion of the line $y = mx$, $m > 0$, that lies in the first quadrant and goes from point (a, ma) to the point (X, mX) , $0 \le a < X$, where a is a constant and X is a variable.

A solid of revolution is obtained by rotating this line through 2π radians about the x-axis and another solid of revolution is obtained by rotating this line through 2π radians about the y-axis.

a If the two volumes are the same for all values of X , use integration to find the value of m. [13]

Now working more generally, consider the portion of the increasing function $y = y(x)$, that lies in the first quadrant and goes from point (a, b) to the point (X, Y) , $0 \le a < X, 0 \le b < Y$. Here a and b are constants and X and Y are variables. A solid of revolution is obtained by rotating this curve through 2π radians about the x-axis and another solid of revolution is obtained by rotating this curve through 2π radians about the y-axis. If the two volumes are the same for all values of X , it can be proved that X and Y must satisfy the differential equation $Y^2 = X^2 \frac{dY}{dX}$

b Solve this differential equation to show that $Y = \frac{X}{1 + cX}$ [6]

It follows that the function $y(x) = \frac{x}{1 + cx}$ between (a, b) and (X, Y) produces a volume of revolution of same magnitude, regardless of which axis it is rotated about.

c Find c in terms of a and b in the function given above. [3]

d Show that the special case $c = 0$ corresponds to the example found in part **a**. [2]

e Use calculus to verify that $y(x) = \frac{x}{1 + cx}$ is indeed an increasing function. [3]

f Investigate the concavity of $y(x) = \frac{x}{1 + cx}$ and the existence of any asymptotes. Consider the necessary different cases, which depend on the value of c. **Remember:** you are only considering the portion of the graph of $y(x) = \frac{x}{1 + cx}$ which lies in the first quadrant, that is $x > 0, y > 0$. [7]

Markscheme

1 a i $\frac{1}{k(k+1)} \equiv \frac{A}{k} + \frac{B}{k+1} \Rightarrow A(k+1) + Bk \equiv 1 + 0k$ M1 A1

Equating coefficients: $A + B = 0, A = 1 \Rightarrow B = -1$ R1 A1

$S_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ M1

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n}\right) - \frac{1}{n+1}$ A1

$= 1 - \frac{1}{n+1}$ A1

$= \frac{n}{n+1}$ AG

ii 1 A1

[8 marks]

b i $\frac{1}{k(k+2)} \equiv \frac{A}{k} + \frac{B}{k+2} \Rightarrow A(k+2) + Bk \equiv 1 + 0k$ M1 A1

Equating coefficients: $A + B = 0,\ 2A = 1 \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$ R1 A1

$T_n = \sum_{k=1}^{n}\frac{\frac{1}{2}}{k} + \frac{\frac{-1}{2}}{k+2} = \left(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{3}\right) + \left(\frac{\frac{1}{2}}{2} - \frac{\frac{1}{2}}{4}\right) + \left(\frac{\frac{1}{2}}{3} - \frac{\frac{1}{2}}{5}\right) + \left(\frac{\frac{1}{2}}{4} - \frac{\frac{1}{2}}{6}\right) \ldots + \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1}\right) + \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}\right)$

M1 A1

$= \frac{1}{2} + \frac{1}{4} + \left(\frac{1}{6} - \frac{1}{6}\right) + \left(\frac{1}{8} - \frac{1}{8}\right) + \ldots + \left(\frac{1}{2n} - \frac{1}{2n}\right) - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$

$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$ A1 M1

$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$ A1

ii $\frac{3}{4}$ A1

[10 marks]

c i $i > i - 1 \Rightarrow \frac{1}{i^2} < \frac{1}{(i-1)i} \Rightarrow \sum_{i=2}^{n}\frac{1}{i^2} < \sum_{i=2}^{n}\frac{1}{(i-1)i}$ R1 R1

$\sum_{i=1}^{n}\frac{1}{i^2} < 1 + \sum_{i=2}^{n}\frac{1}{(i-1)i} \Rightarrow Q_n < 1 + \sum_{k=1}^{n-1}\frac{1}{k(k+1)}$ M1 A1

So $Q_n < 1 + S_{n-1} = 1 + \frac{n-1}{n} = \frac{2n-1}{n}$ R1 AG

ii Addition of the first three terms of Q_n gives $1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$ M1 A1

$\lim_{n\to\infty}\frac{2n-1}{n} = 2$ A1

So $\frac{49}{36} < Q \le 2$ AG

[8 marks]

[Total: 26 marks]

2 a $\pi\int_{a}^{X} y^2 dx = \pi\int_{ma}^{mX} x^2 dy$ M1 A1 A1

$\int_{a}^{X} m^2x^2 dx = \int_{ma}^{mX}\frac{y^2}{m^2} dy$ M1 A1 A1

$$\left[\frac{m^2x^3}{3}\right]_a^X = \left[\frac{y^3}{3m^2}\right]_{ma}^{mX}$$ A1 A1

$$\frac{m^2X^3}{3} - \frac{m^2a^3}{3} = \frac{m^3X^3}{3m^2} - \frac{m^3a^3}{3m^2}$$ A1 A1

$$m^2\left(X^3 - a^3\right) = m\left(X^3 - a^3\right)$$ A1

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 1$$ M1 A1

[13 marks]

b $\int \frac{1}{X^2}\,dX = \int \frac{1}{Y^2}\,dY$ M1 A1

$$\frac{-1}{X} = \frac{-1}{Y} + c$$ A1 A1

$$\frac{1}{Y} = \frac{1}{X} + c = \frac{1+cX}{X}$$ M1 A1

$$Y = \frac{X}{1+cX}$$ AG

[6 marks]

c $b = \frac{a}{1+ca} \Rightarrow b + cab = a \Rightarrow cab = a - b$ R1 M1

$$c = \frac{a-b}{ab}$$ A1

[3 marks]

d $c = 0$ gives $y = x$ and $a = b$ which corresponds to part **a** with $m = 1$ R1 R1

[2 marks]

e $\frac{dy}{dx} = \frac{(1+cx) - xc}{(1+cx)^2} = \frac{1}{(1+cx)^2}$ M1 A1

This is always positive, confirming that the function is increasing. R1

[3 marks]

f $\frac{d^2y}{dx^2} = \frac{-2c}{(1+cx)^3}$ M1 A1

Since the curve is in the first quadrant, $y, x > 0$ and since $y(x) = \frac{x}{1+cx}$, it follows that

$1 + cx > 0$ R1

If $c > 0$, $\frac{d^2y}{dx^2} < 0$ and hence $y(x) = \frac{x}{1+cx}$ is concave down, R1

with horizontal asymptote $y = \frac{1}{c}$ A1

If $c < 0$, $\frac{d^2y}{dx^2} > 0$ and hence $y(x) = \frac{x}{1+cx}$ is concave up, R1

with horizontal asymptote $x = \frac{-1}{c}$ A1

If $c = 0$, it is the straight line given in part **a**.

[7 marks]

[Total: 34 marks]